The Incomplete Gamma Function
Part II - Base Equation For A Mean-Reverting Process

Gary Schurman, MBE, CFA

December, 2017

In this white paper we will build the base equation for a mean-reverting processes. To assist us in this endeavor we will solve the following hypothetical problem...

Our Hypothetical Problem

Given the following base equation (as defined below) parameters...

$$a = 0.60 \;\text{...and...}\; b = 0.40 \;\text{...and...}\; c = -0.10 \;\text{...and...}\; d = 0.20 \tag{1}$$

Answer the following questions...

Question 1: Graph the integrand over the time interval $[0, 20]$.

Question 2: What is the area beneath the curve from $t = 2$ to $t = 8$?

The Base Equation

We will define the function $f(t)$ to be an integral in the following form...

$$f(t) = \int\limits_m^n \mathrm{Exp}\left\{d + c\,t - a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t \;\text{...where...}\; 0 < m < n \tag{2}$$

The parameters in Equation (2) above can be interpreted as follows...

Param	Description
a	Short-term rate (unsustainable) minus the long-term rate (sustainable)
b	Rate of mean reversion
c	Long-term rate (sustainable)
d	A catch all constant
t	Time in years

Since the variable d in Equation (2) above is not a function of time then we can remove that variable from within the integral and rewrite that equation as...

$$f(t) = \mathrm{Exp}\left\{d\right\}\int\limits_m^n \mathrm{Exp}\left\{c\,t - a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t \tag{3}$$

We will define the variable I to be the integral that we must solve. Using Equation (3) above the equation for the integral to be solved is...

$$I = \int\limits_m^n \mathrm{Exp}\left\{c\,t - a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t \tag{4}$$

Parameter Constraints

We will define the function $g(t)$ to be the integrand of the integral as defined by Equation (4) above. The equation for the function $g(t)$ is...

$$g(t) = \text{Exp}\left\{c\,t - a\,\text{Exp}\left\{-\,b\,t\right\}\right\} = \text{Exp}\left\{c\,t\right\}\text{Exp}\left\{-\,a\,\text{Exp}\left\{-\,b\,t\right\}\right\}\Bigg] \tag{5}$$

Our goal is to solve the integral in Equation (4) above and therefore we have to prove that the parameter values a, b and c are such that the integrand converges to zero as time goes to infinity. Using Equation (5) above we want to set the parameters such that we achieve the following limit...

$$\text{we want to prove that...}\ \lim_{t\to\infty} g(t) = 0 \tag{6}$$

As time goes to infinity the first term in Equation (5) above goes to zero. This statement in equation form is...

$$\lim_{t\to\infty}\text{Exp}\left\{c\,t\right\} = 0\ \text{...when...}\ c < 0 \tag{7}$$

As time goes to infinity the exponential within the second term in Equation (5) above goes to zero. This statement in equation form is...

$$\lim_{t\to\infty} -a\,\text{Exp}\left\{-\,b\,t\right\} = 0\ \text{...when...}\ b > 0 \tag{8}$$

Using Equation (8) above the second term in Equation (5) above goes to one as time goes to infinity. This statement in equation form is...

$$\lim_{t\to\infty}\text{Exp}\left\{-\,a\,\text{Exp}\left\{-\,b\,t\right\}\right\} = \text{Exp}\left\{0\right\} = 1 \tag{9}$$

If we combine Equations (7) and (9) above then the limit of Equation (5) as time goes to infinity is zero, which is what we wanted to prove. This statement in equation form is...

$$\lim_{t\to\infty}\text{Exp}\left\{c\,t - a\,\text{Exp}\left\{-\,b\,t\right\}\right\} = 0\times 1 = 0 \tag{10}$$

To be consistent with an integrand that converges to zero and a mean-reverting function we will define the following parameter constraints...

$$a > 0\ \text{...and...}\ b > 0\ \text{...and...}\ c < 0 \tag{11}$$

Note that the equation for the derivative of Equation (5) above with respect to time is...

$$\frac{\delta g(t)}{\delta t} = \left(c + a\,b\,\text{Exp}\left\{-\,b\,t\right\}\right)\times g(t) \tag{12}$$

Using derivative Equation (12) and the limit calculations above we can make the following statement...

$$\begin{aligned}&\text{when...}\ a\,b\,\text{Exp}\left\{-b\,t\right\} > |c|\ \text{...then the integrand is increasing}\\&\text{when...}\ a\,b\,\text{Exp}\left\{-b\,t\right\} < |c|\ \text{...then the integrand is decreasing}\end{aligned} \tag{13}$$

Note that as time goes to infinity...

$$\text{then}\ a\,b\,\text{Exp}\left\{-b\,t\right\} < |c|\ \text{so the integrand is decreasing} \tag{14}$$

What we may observe in practice is that the integrand increases for small values of t, reaches a plateau, and then decreases for larger values of t, and equals zero at time infinity.

The First Change Of Variables

We will make the following definitions...

$$\text{if...}\ u^{-b} = \text{Exp}\left\{\ln(u^{-b})\right\} = \text{Exp}\left\{-\,b\,\ln(u)\right\}\ \text{...then...}\ t = \ln(u)\ \text{...such that...}\ u = \text{Exp}\left\{t\right\} \tag{15}$$

The derivative of Equation (15) above with respect to the variable t is...

$$\text{if...}\;\; u=\mathrm{Exp}\left\{t\right\}\;\;\text{...then...}\;\;\frac{\delta u}{\delta t}=\mathrm{Exp}\left\{t\right\}\;\;\text{...such that...}\;\;\delta u=\mathrm{Exp}\left\{t\right\}\delta t=u\,\delta t \tag{16}$$

Using Equations (15) and (16) above we can rewrite Equation (**??**) above as...

$$I=\int\limits_{t=m}^{t=n}\mathrm{Exp}\left\{c\,\ln(u)-a\,u^{-b}\right\}u^{-1}\,u\,\delta t=\int\limits_{t=m}^{t=n}\mathrm{Exp}\left\{c\,\ln(u)\right\}\mathrm{Exp}\left\{-\,a\,u^{-b}\right\}u^{-1}\,u\,\delta t \tag{17}$$

Using Equation (15) above we will change the bounds of integration as follows...

$$\bar{m}=\mathrm{Exp}\left\{m\right\}\;\;\text{...and...}\;\;\bar{n}=\mathrm{Exp}\left\{n\right\}\;\;\text{...since...}\;\;u=\mathrm{Exp}\left\{t\right\} \tag{18}$$

Using Equations (16) and (18) above we can rewrite Equation (17) above as...

$$I=\int\limits_{u=\bar{m}}^{u=\bar{n}}u^{c-1}\,\mathrm{Exp}\left\{-\,a\,u^{-b}\right\}\delta u \tag{19}$$

The Second Change Of Variables

We will make the following definitions...

$$v=u^{c}a^{-\frac{c}{b}}\;\;\text{...where...}\;\;\frac{\delta v}{\delta u}=c\,u^{c-1}a^{-\frac{c}{b}}\;\;\text{...such that...}\;\;\delta v=u^{c-1}a^{-\frac{c}{b}}\,c\,\delta u \tag{20}$$

Using Equation (20) above note the following...

$$\text{if...}\;\; v=u^{c}a^{-\frac{c}{b}}\;\;\text{...then...}\;\;a^{\frac{c}{b}}v=u^{c}\;\;\text{...and...}\;\;(a^{\frac{c}{b}}v)^{\frac{1}{c}}=(u^{c})^{\frac{1}{c}}\;\;\text{...and...}\;\;u=a^{\frac{1}{b}}v^{\frac{1}{c}} \tag{21}$$

Using Equations (20) and (21) above we can rewrite Equation (19) above as...

$$I=\int\limits_{v=\bar{m}}^{v=\bar{n}}u^{c-1}\,\mathrm{Exp}\left\{-\,a\times(a^{\frac{1}{b}}v^{\frac{1}{c}})^{-b}\right\}\delta u=\int\limits_{v=\bar{m}}^{v=\bar{n}}u^{c-1}\,\mathrm{Exp}\left\{-\,v^{-\frac{b}{c}}\right\}\delta u \tag{22}$$

Multiply both sides of Equation (22) above by a constant...

$$I=a^{\frac{c}{b}}c^{-1}\int\limits_{v=\bar{m}}^{v=\bar{n}}\mathrm{Exp}\left\{-\,v^{-\frac{b}{c}}\right\}u^{c-1}\,a^{-\frac{c}{b}}\,c\,\delta u \tag{23}$$

Using Equations (18) and (20) above we will change the bounds of integration as follows...

$$\hat{m}=\bar{m}^{c}a^{-\frac{c}{b}}=\mathrm{Exp}\left\{c\,m\right\}a^{-\frac{c}{b}}\;\;\text{...and...}\;\;\hat{n}=\bar{n}^{c}a^{-\frac{c}{b}}=\mathrm{Exp}\left\{c\,n\right\}a^{-\frac{c}{b}}\;\;\text{...since...}\;\;v=u^{c}a^{-\frac{c}{b}} \tag{24}$$

Using Equations (23) and (24) above we can rewrite Equation (22) above as...

$$I=a^{\frac{c}{b}}c^{-1}\int\limits_{v=\hat{m}}^{v=\hat{n}}\mathrm{Exp}\left\{-\,v^{-\frac{b}{c}}\right\}\delta v \tag{25}$$

The Third Change Of Variables

We will make the following definitions...

$$w=v^{-\frac{b}{c}}\;\;\text{...where...}\;\;\frac{\delta w}{\delta v}=-\frac{b}{c}\,v^{(-\frac{b}{c}-1)}\;\;\text{...such that...}\;\;\delta w=-\frac{b}{c}\,v^{-\frac{b+c}{c}}\,\delta v \tag{26}$$

Using Equation (26) above note the following...

$$\text{if... } w = v^{-\frac{b}{c}} \text{ ...then... } v = w^{-\frac{c}{b}} \text{ ...and... } \delta w = -\frac{b}{c}\,(w^{-\frac{c}{b}})^{-\frac{b+c}{c}}\,\delta v = -\frac{b}{c}\,w^{\frac{c}{b}+1}\,\delta v \tag{27}$$

Using Equations (26) and (27) above (and multiplying by a constant and the constant's reciprcal) we can rewrite Equation (25) above as...

$$I = a^{\frac{c}{b}}\,c^{-1}\int\limits_{v=\hat{m}}^{v=\hat{n}} \mathrm{Exp}\left\{-\,v^{-\frac{b}{c}}\right\} \times -\frac{c}{b}\,w^{-\frac{c}{b}-1} \times -\frac{b}{c}\,w^{\frac{c}{b}+1} \times \delta v \tag{28}$$

Using Equations (24) and (26) above we will change the bounds of integration as follows...

$$\begin{aligned} m* &= \hat{m}^{-\frac{b}{c}} = (\bar{m}^c a^{-\frac{c}{b}})^{-\frac{b}{c}} = \bar{m}^{-b} a = a\,\mathrm{Exp}\left\{-\,b\,m\right\} \text{ ...and...} \\ n* &= \hat{n}^{-\frac{b}{c}} = (\bar{n}^c a^{-\frac{c}{b}})^{-\frac{b}{c}} = a\,\bar{n}^{-b} a = \mathrm{Exp}\left\{-\,b\,n\right\} \text{ ...since... } w = v^{-\frac{b}{c}} \end{aligned} \tag{29}$$

Note that since the parameters a and b are both greater than zero then $n*$, which is the upper bound of integration, is less than $m*$, which is the lower bound of integration. For this reason will will want to switch the bounds of integration. Using Equation (29) above we can rewrite Equation (28) above as...

$$\begin{aligned} I &= a^{\frac{c}{b}}\,c^{-1}\int\limits_{v=\hat{m}}^{v=\hat{n}} \mathrm{Exp}\left\{-\,w\right\} \times -\frac{c}{b}\,w^{-\frac{c}{b}-1}\,\delta w \\ &= -a^{\frac{c}{b}}\,b^{-1}\int\limits_{w=m*}^{w=n*} w^{(-\frac{c}{b})-1}\,\mathrm{Exp}\left\{-\,w\right\}\delta w \\ &= a^{\frac{c}{b}}\,b^{-1}\int\limits_{w=n*}^{w=m*} w^{(-\frac{c}{b})-1}\,\mathrm{Exp}\left\{-\,w\right\}\delta w \end{aligned} \tag{30}$$

The Solution To The Base Equation

Using Equations (3) and (30) above the solution to Equation (2) above is...

$$\begin{aligned} &f(t) = \int\limits_{m}^{n} \mathrm{Exp}\left\{d + c\,t - a\,\mathrm{Exp}\left\{-\,b\,t\right\}\right\}\delta t = \mathrm{Exp}\left\{d\right\} a^{\frac{c}{b}}\,b^{-1}\int\limits_{n*}^{m*} w^{\alpha-1}\,\mathrm{Exp}\left\{-\,w\right\}\delta w \\ &\text{where... } \alpha = -\frac{c}{b} \text{ ...and... } m* = a\,\mathrm{Exp}\left\{-\,b\,m\right\} \text{ ...and... n}^* = a\,\mathrm{Exp}\left\{-\,b\,n\right\} \end{aligned} \tag{31}$$

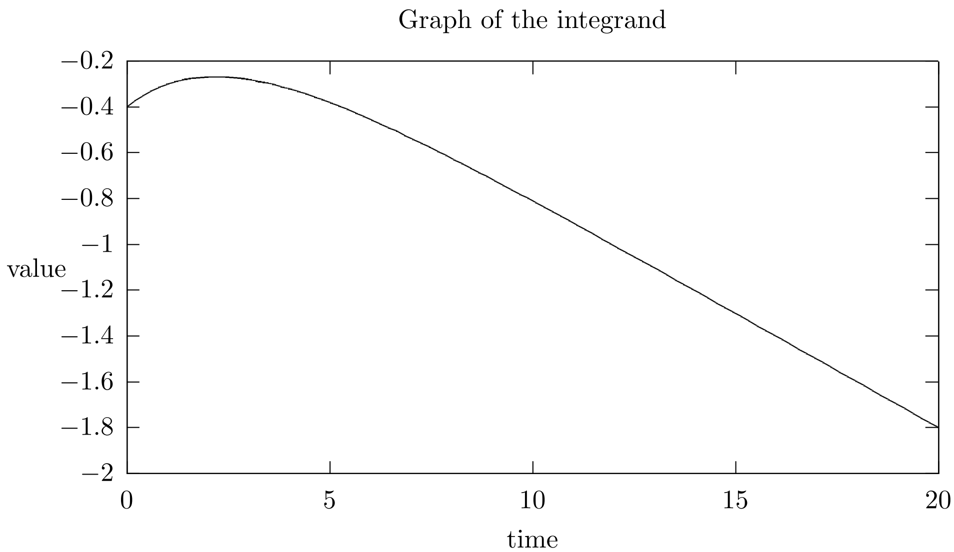
Note that the integral in Equation (31) above is the upper incomplete gamma function and therefore we can rewrite that equation as...

$$\begin{aligned} f(t) &= \mathrm{Exp}\left\{d\right\} a^{-\alpha} b^{-1}\Big(\Gamma(\alpha, n*) - \Gamma(\alpha, m*)\Big) \\ &= \mathrm{Exp}\left\{d\right\} a^{\frac{c}{b}}\,b^{-1}\left[\Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,n\right\}\right) - \Gamma\left(-\frac{c}{b},\,a\,\mathrm{Exp}\left\{-\,b\,m\right\}\right)\right] \end{aligned} \tag{32}$$

The Answers To Our Hypothetical Problem

Question 1: Graph the integrand over the time interval $[0, 20]$.

$$\text{integrand} = \text{Exp}\left\{d + c\,t - a\,\text{Exp}\left\{-\,b\,t\right\}\right\} \tag{33}$$



Question 2: What is the area beneath the curve from $t = 2$ (lower bound m) to $t = 8$ (upper bound n)?

Using Equation (32) above the solution to our base equation, which is the area under the curve, is...

$$f(t) = \text{Exp}\left\{d\right\} a^{\frac{c}{b}} b^{-1}\left[\Gamma\left(-\frac{c}{b}, a\,\text{Exp}\left\{-\,b\,n\right\}\right) - \Gamma\left(-\frac{c}{b}, a\,\text{Exp}\left\{-\,b\,m\right\}\right)\right] \tag{34}$$

Using the parameters to our problem (Equation (1) above)...

$$\text{Exp}\left\{d\right\} = \text{Exp}\left\{0.20\right\} = 1.2214 \ \text{...and...}\ a^{\frac{c}{b}} b^{-1} = 0.60^{-\frac{0.10}{0.40}} \times 0.40^{-1} = 2.8405 \tag{35}$$

Using Appendix Equation (39) below the solution to the upper incomplete gamma function where $n = 8$ is...

$$\Gamma\left(-\frac{-0.10}{0.40}, 0.60 \times \text{Exp}\left\{-\,0.40 \times 8\right\}\right) = \Gamma\left(0.25, 0.0245\right) = 2.0515 \tag{36}$$

Using Appendix Equation (39) below the solution to the upper incomplete gamma function where $m = 2$ is...

$$\Gamma\left(-\frac{-0.10}{0.40}, 0.60 \times \text{Exp}\left\{-\,0.40 \times 2\right\}\right) = \Gamma\left(0.25, 0.2696\right) = 0.8878 \tag{37}$$

Using Equations (34), (35), (36) and (37) above the area under the curve is...

$$f(t) = 1.2214 \times 2.8405 \times \left(2.0515 - 0.8878\right) = 4.0373 \tag{38}$$

References

[1] Gary Schurman, *The Incomplete Gamma Function - Part I,* December, 2017

Appendix

A. The solution to the upper incomplete gamma function from Part I using standard Excel functions is... [1]

$$\Gamma(\alpha, x) = \text{EXP(GAMMALN(alpha))} \times (1 - \text{GAMMA.DIST(x,alpha,1,true)}) \tag{39}$$